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ABSTRACT

An algorithm is presented that finds the convex hull of any orderly sequence of points in linear time. It is then shown that any simple polygon is an orderly sequence. Hence, the algorithm constructs the convex hull of any simple polygon in linear time.

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1. Introduction

Given a finite set R of points on the plane, the convex hull of R is the convex set that has the smallest area and contains every point of R. Many different $O(n \log n)$ time algorithms exist that find the convex hull of a set of n points [1,3,6,7]. These algorithms are optimal because construction of the convex hull of a set of n points has an $\Omega(n \log n)$ lower bound [7,10].

A polygon is a closed curve that consists of a finite number of line segments. The line segments are the edges of the polygon and their endpoints are the vertices of the polygon. A simple polygon is a polygon which is a simple closed curve. When a polygon is simple, the polygon refers to the set of points of the closed region bounded by the simple closed curve. A simple polygon is convex if it is a convex set.

The convex hull of a polygon is the convex hull of its vertex set. Hence, each of the above algorithms finds the convex hull of a polygon. Unlike sets of points, however, polygons, and simple polygons in particular, are highly structured. Taking advantage of this, Sklansky in [8] presented an algorithm that finds the convex hull of a simple polygon in $O(n)$ time. Recently in [2], it was illustrated that the algorithm of Sklansky does not find the convex hull of every simple polygon.

An O(n) time algorithm is given in [5] that uses two stacks and finds the convex hull of a simple polygon. More recently in [4], an algorithm that uses only one stack was presented.

Because of its simplicity and elegance, an interesting subclass of simple polygons was determined for which Sklansky's algorithm works. It was shown in [9] that the algorithm finds the convex hull of simple polygons which are weakly externally visible.

Our question is whether or not there exists an algorithm with the simplicity of Sklansky's that does not use any stack and still finds the convex hull of a simple polygon. *In this paper we answer* the question by developing such an algorithm. In fact, the algorithm is a slight modification of Sklansky's algorithm. We present an algorithm that constructs the convex hull of a polygon, called an orderly polygon, in linear time. We then show that every simple polygon is an orderly polygon. Thus, we have the desired algorithm.

2. Convex hull of simple polygons

Given any two points z, z' on the plane, $\overline{zz'}$ denotes the line segment between z and z' and $\overrightarrow{zz'}$ the line passing through z and z' with its direction from z to z'.

Let $Q=(q_1,q_2,\cdots,q_n)$ be a sequence of points on the plane, that is, $q_i=(x_i,y_i)$. We identify Q with a polygon whose vertices are the q_i's and edges are the $\overline{q_iq_{i+1}}$'s for all i, $1\leq i\leq n$, where i+1=1 when i=n. From now on we use the terms, sequence of points and polygon, interchangeably. The convex hull H(Q) of the points of Q is a sequence of some points of Q such that as a polygon, it is convex and contains every point of Q. A sequence Q is said to be an <u>orderly sequence</u> (or <u>orderly polygon</u>) if its convex hull H(Q) is a subsequence of Q in clockwise order. That is, if $H(Q)=(q_{i_1},q_{i_2},\cdots,q_{i_m})$, then $i_1<i_2<\cdots<i_m$ and H(Q) lies to the right of $\overrightarrow{q_{i_1}q_{i_2}}$. In Figure 1, the sequence of points

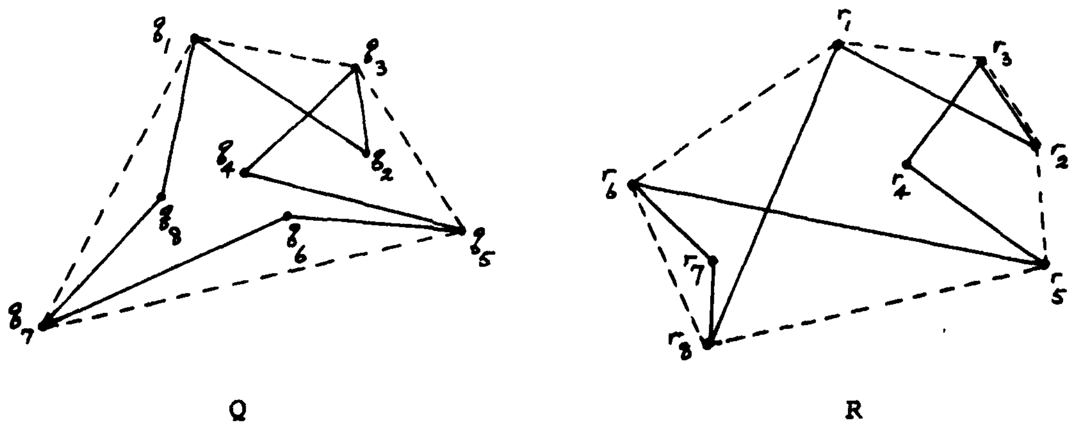

Figure 1. Two sequences and their convex hulls.

$Q=(q_1,\cdots,q_8)$ is an orderly sequence but $R=(r_1,\cdots,r_8)$ is not.

Given a sequence $Q=(q_1,q_2,\cdots,q_n)$ of points, let ℓ_x, ℓ_x', ℓ_y and ℓ_y' be the upper horizontal, lower horizontal, left vertical and right vertical supports of the polygon Q, respectively. Let q_{e_1} be the leftmost point of Q on ℓ_x, that is, $y_{e_1} \geq y_i$ for all i, $1 \leq i \leq n$, and if $y_{e_1} = y_i$ then $x_{e_1} < x_i$. Without loss of generality assume that $q_{e_1} = q_1$. Let q_{e_2} be the highest point of Q on ℓ_y', q_{e_3} the rightmost point of Q on ℓ_x' and q_{e_4} the lowest point of Q on ℓ_y. (See Figure 2.)

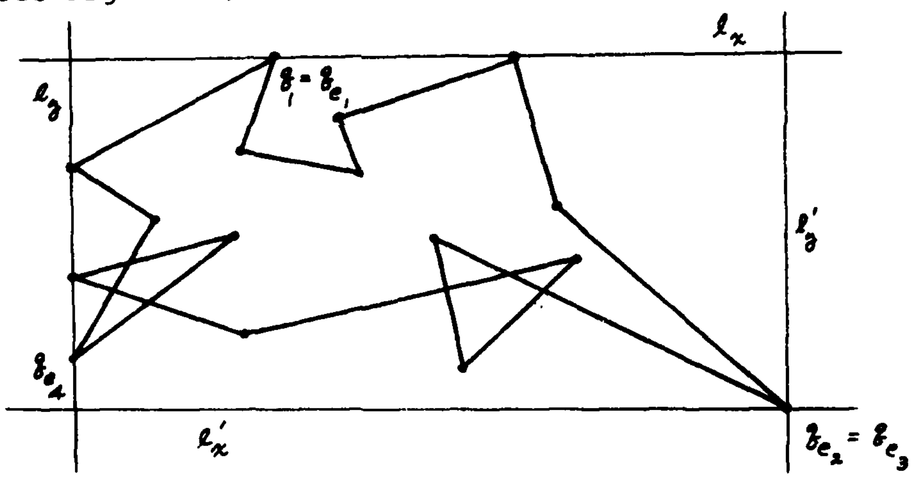

Figure 2. A polygon Q and its extreme points.

These points $q_{e_1},\cdots,q_{e_4}$ are called the extreme points of Q. We note that the q_{e_i}'s are not necessarily distinct. But we assume that a polygon is not degenerate, that is, not all vertices are collinear. Thus, a polygon has at least three distinct extreme points.

Next we derive a few preliminary results that lead to the development of a linear time algorithm to find the convex hull of orderly polygons. Two obvious facts are first stated as lemmas without proof.

<u>Lemma 1</u>

Every extreme point of Q is a vertex of the convex hull $H(Q)$ of Q.

<u>Lemma 2</u>

If Q is an orderly sequence, then $e_i \le e_{i+1}$ for all i, $1 \le i \le 4$, where $e_1=1$.

For $q_i, q_j, q_k \in Q$, $\angle q_i q_j q_k$ is the counterclockwise angle from $\overline{q_i q_j}$ to $\overline{q_j q_k}$. We note that $\angle q_i q_j q_k$ is convex if and only if $d=(y_j-y_i)(x_k-x_j)-(y_k-y_j)(x_j-x_i)>0$. Hence, we check the sign of d to determine whether or not $\angle q_i q_j q_k$ is convex.

<u>Lemma 3</u>

Let $Q=(q_1,\cdots,q_{e_i},\cdots,q_k,\cdots,q_{e_{i+1}},\cdots,q_n)$ be an orderly polygon. If $\angle q_{j-1} q_j q_{j+1}$ is convex for all j, $e_i \le j < k$ and q_k is a vertex of H(Q), then each q_j, $e_i \le j < k$, is a vertex of H(Q).

<u>Proof</u>: Since Q is orderly, H(Q) is a subsequence of Q. By Lemma 1, q_{e_i} and $q_{e_{i+1}}$ are vertices of H(Q). Suppose that not all of the q_j's, $e_i \le j < k$, are vertices of H(Q).

Let u, v be such that $e_i \le u < v \le k$ and q_j is a vertex of H(Q) if $e_i \le j \le u$, q_h is not a vertex of H(Q) if $u < h < v$, and q_v is a vertex of H(Q). Then $\overline{q_u q_v}$ is an edge of H(Q) and every point of Q lies to the right of $\overrightarrow{q_u q_v}$. But since they are convex vertices, each q_j, $u<j<v$, lies to the left of $\overrightarrow{q_u q_v}$, which is a contradiction. □

Lemma 4

Let Q be an orderly sequence and k an integer such that $e_i < k < e_{i+1}$ for some i, $1 \le i \le 4$ where $i+1=1$ when $i=4$. If $\angle q_{j-1} q_j q_{j+1}$ is convex for all j, $e_i \le j < k$, and $\angle q_{k-1} q_k q_{e_{i+1}}$ is not convex, then q_k is not a vertex of H(Q).

Proof: Suppose that q_k is a vertex of H(Q). Since $\angle q_{k-1} q_k q_{e_{i+1}}$ is concave, $q_{e_{i+1}}$ lies either on $\overrightarrow{q_{k-1} q_k}$ or to its left. By Lemma 3, q_{k-1} is also a vertex of H(Q) and every point of Q lies to the right of $\overrightarrow{q_{k-1} q_k}$. Thus, $q_{e_{i+1}}$ must lie on $\overrightarrow{q_{k-1} q_k}$. Then $\overline{q_{k-1} q_{e_{i+1}}}$ is an edge of H(Q) and q_k which is a point on $\overline{q_{k-1} q_{e_{i+1}}}$ is not a vertex of H(Q). This is a contradiction. □

Lemma 5

Let Q be an orderly sequence and k an integer such that $e_i < k < e_{i+1}$ for some i, $1 \le i \le 4$ where $i+1=1$ when $i=4$. If $\angle q_{j-1} q_j q_{j+1}$ is convex for all j, $e_i \le j < k$, and $\angle q_{k-1} q_k q_{k+1}$ is not convex, then q_k is not a vertex of H(Q).

Proof: Similar to the one for the above lemma and omitted. □

We note that the assertions of Lemmas 4 and 5 do not hold if either Q is not orderly or $<q_{j-1}q_jq_{j+1}$ is concave for some j, $e_i<j<k$. For example, consider R in Figure 1, which is not an orderly sequence. Both $<r_1r_2r_3$ and $<r_5r_6r_8$ $(= <r_5r_6r_{e_3})$ are concave but r_2 and r_6 are both vertices of H(R). Next consider an orderly sequence Q in Figure 3. Again both $<q_3q_4q_7 (= <q_3q_4q_{e_2})$ and $<q_3q_4q_5$ are concave, but

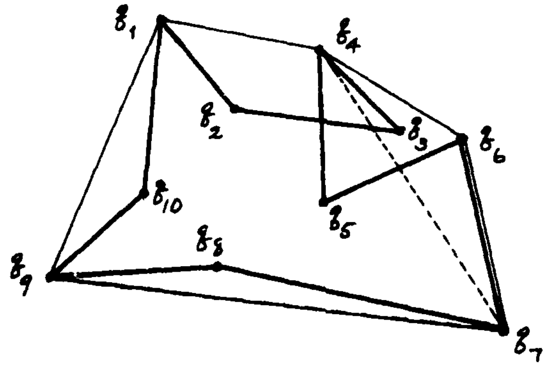

Figure 3. An orderly sequence Q.

q_4 is a vertex of H(Q).

Now we present the algorithm that finds the convex hull of orderly polygons (CHOP). Note that since a sequence Q is an orderly sequence, H(Q) may be obtained, without re-ordering of elements, by removing elements of Q that are not vertices of H(Q). Informally, the algorithm finds the extreme points $q_{e_1},\cdots,q_{e_4}$ each of which is a vertex of H(Q). Then the algorithm removes from Q every point q_k, $e_i<k<e_{i+1}$, $1\le i\le 4$, that cannot be a vertex of H(Q). In the algorithm, given a point q_k of Q, q_{k-1} and q_{k+1} represent the predecessor

and successor of q_k among the current sequence of points of Q.

<u>Algorithm</u> CHOP(Q)

1. Find the extreme points of Q, $q_1=q_{e_1}, q_{e_2}, q_{e_3}$, and q_{e_4}.

2. [Initialization and termination]

2.1 Set $i \leftarrow 1$

2.2 <u>If</u> $i>4$ <u>then</u> stop.

<u>If</u> $q_{e_i}=q_{e_{i+1}}$ <u>then</u> set $i \leftarrow i+1$; go to step 2.2.

Set $q_k \leftarrow q_{e_i+1}$.

2.3 If $q_k=q_{e_{i+1}}$ <u>then</u> set $i \leftarrow i+1$; <u>go</u> <u>to</u> step 2.2.

3. [Remove from Q points that are not vertices of H(Q).]

3.1 <u>If</u> $\angle q_{k-1} q_k q_{e_{i+1}}$ is concave

    <u>then</u> remove q_k from Q; set $q_k \leftarrow q_{k+1}$; <u>go</u> <u>to</u> step 2.3.

3.2 <u>If</u> $\angle q_{k-1} q_k q_{k+1}$ is concave

    <u>then</u> remove q_k from Q

        <u>if</u> $q_{k-1}=q_{e_i}$ <u>then</u> set $q_k \leftarrow q_{k+1}$; <u>go</u> <u>to</u> step 2.3.

        <u>else</u> set $q_k \leftarrow q_{k-1}$; <u>go</u> <u>to</u> step 3.1.

    <u>else</u> set $q_k \leftarrow q_{k+1}$; <u>go</u> <u>to</u> step 2.3.

Theorem 6

The algorithm CHOP finds the convex hull of an orderly sequence of points in time linear in n, where n is the number of points of Q.

Proof: It is immediate that the algorithm runs in time linear in the number of points of the sequence.

We must show that the algorithm in fact finds the convex hull of any orderly sequence of points $Q=(q_1,q_2,\cdots,q_n)$. We claim the following:

(i) When the algorithm checks in steps 3.1 and 3.2 whether a point q_k, $e_i<k<e_{i+1}$ for some $1\leq i\leq 4$, must be removed from Q, $<q_{j-1}q_jq_{j+1}$ is convex for all j, $e_i\leq j<k$.

(ii) When a point q_k is removed from an orderly sequence during the execution of the algorithm, the resulting sequence is still orderly.

(iii) Let Q be the resulting sequence at the termination of the algorithm. Then Q is a simple polygon and every point of Q is convex.

Then by Lemmas 4 and 5, (i) and (ii) guarantee that the algorithm removes only the vertices that are not vertices of H(Q). By (iii), Q at the termination of the algorithm is a simple convex polygon and therefore, is the convex hull H(Q) of the original orderly sequence Q.

We now prove our claims:

(i) Initially, when $q_k = q_{e_i+1}$ for some i, $1 \le i \le 4$, claim (i) is true, since $\langle q_{e_i-1} q_{e_i} q_k$ is convex. Subsequently, for all j, $e_i \le j < k$, $\langle q_{j-1} q_j q_{j+1}$ is kept convex because q_k is set to q_{k+1} only if $\langle q_{k-1} q_k q_{k+1}$ is convex and the algorithm backtracks otherwise.

(ii) Let $Q=(q_1,\cdots,q_k,\cdots,q_n)$ be an orderly sequence and suppose that q_k is removed from Q in step 3.1 or 3.2. Then by Lemma 4 or 5, respectively, q_k is not a vertex of H(Q). Thus the convex hull H(Q') of $Q'=(q_1,\cdots,q_{k-1},q_{k+1},\cdots,q_n)$ is the same polygon as H(Q). Since H(Q) is a subsequence of Q and q_k is not an element of H(Q), H(Q) is also a subsequence of Q'. Since H(Q')=H(Q), H(Q') is a subsequence of Q' and Q' is an orderly sequence.

(iii) Step 3.1 of the algorithm prevents the cumulative change of direction of directed edges of Q between q_{e_i} and $q_{e_{i+1}}$ from exceeding $\pi/2$ for each i, $1 \le i \le 4$. Therefore, no edge can cross any other edge and Q is a simple polygon. Also because of (i), at the termination of the algorithm, every point of Q is convex. □

We have presented an algorithm that finds the convex hull of an orderly sequence in linear time. However, the main objective of this paper is to develop a linear time algorithm to find the convex hull of simple polygons. Next we show that simple polygons are orderly sequences. Therefore, the algorithm CHOP finds the convex hull of any simple polygon.

Lemma 7

If $Q=(q_1,q_2,\cdots,q_n)$ is a simple polygon such that the polygon lies to the right of directed edge from q_i to q_{i+1} for each i, $1\le i\le n$, then it is an orderly polygon.

Proof: Suppose the contrary. Then there exists a simple polygon which is not orderly. Let $Q=(q_1,q_2,\cdots,q_n)$ be such a simple polygon and $H(Q)=(q_{i_1},\cdots,q_{i_m})$, where $q_1=q_{e_1}=q_{i_1}$. Suppose that j is an integer such that $i_1<i_2<\cdots<i_j$ and $i_j>i_{j+1}$. If m=3, then $H(Q)=(q_{i_1},q_{i_2},q_{i_3})$ and $1<i_2$ and $i_2>i_3$. Let $P_{1,3}$ be the path $(q_1,q_2,\cdots,q_{i_3})$ in Q. Then $P_{1,3}$ lies within H(Q) and the polygon Q lies to the right of $P_{1,3}$. Let $P_{3,2}$ be the path $(q_{i_3},\cdots,q_{i_2})$ in Q. Then $P_{3,2}$ must lie within H(Q) and to the right of $P_{1,3}$. But this is impossible. (See Figure 4(a)). Now suppose that $m\ge 4$.

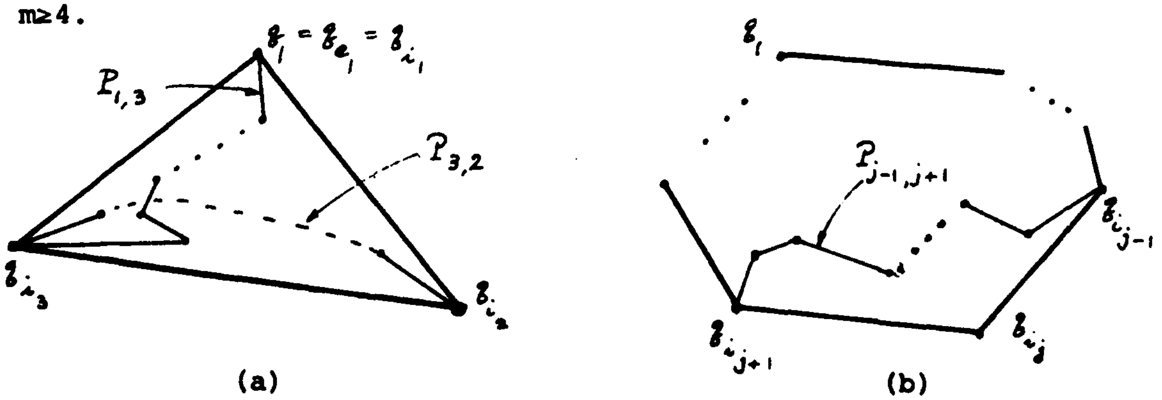

Figure 4.

Let $P_{j-1,j+1}$ be the path $(q_{i_{j-1}},\cdots,q_{i_{j+1}})$ in Q. (We assume that $i_{j-1}<i_{j+1}$. Otherwise consider the path $P_{j+1,j-1}=(q_{i_{j+1}},\cdots,q_{i_{j-1}})$ in Q.) Then $P_{j-1,j+1}$ partitions H(Q) and separates q_{i_j} from q_{i_k} for some k such that the path P_{jk} (or P_{kj}) in Q does not contain either $q_{i_{j-1}}$ or $q_{i_{j+1}}$. The path lies within H(Q) and thus must cross $P_{j-1,j+1}$, which is a contradiction. (See Figure 4(b)). Therefore, P must be an orderly polygon. $\square$

We state the main result of the paper as a theorem:

<u>Theorem 8</u>

The algorithm CHOP finds the convex polygon of any simple polygon in time linear in the number of vertices of the polygon.

3. Conclusions

A linear time algorithm was presented that constructs the convex hull of simple polygons. It is as simple as Sklansky's algorithm and does not require any stack. Moreover, the class of polygons for which the algorithm works contains properly the class of simple polygons.

Although a polygon, which is a sequence of points, is more structured than a set of points, $\Omega(n \log n)$ is still the lower bound for construction of the convex hull of a polygon in general. The algorithm presented in this paper can be applied to orderly sequences only. Besides the class of simple polygons, we have not been able to identify any subclass of orderly polygons that is encountered in applications.

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